Applicationof Heat Equationby Finite Difference Methods

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Abstract

In this paper, the forward-difference formula, the backward-difference formula and the central-difference formula are studied. Firstly, the explicit formula of one finite-difference approximation to heat equation is derived. Then we calculate the numerical solutions of heat equation by using Matlab programming. We also discuss the Crank-Nicolson implicit formula. Finally, the solution of the second-order parabolic equation with initial-boundary conditions is derived by using Crank-Nicolson implicit method.

Key words: finite-difference, explicit, implicit.

Introduction

Three basic types of partial differential equations are distinguished: parabolic, hyperbolic and elliptic. The solution of the equation pertaining to each of the types has their own characteristic qualitative differences.

Finite-difference approximations to derivatives

Assume that U is a function of the independent variables x and t. Subdivide the x-t plane into sets of equal rectangles of sides $\delta x = h$, $\delta t = k$, by equally spaced grid lines parallel to OY, defined by $x_i = ih$, i = 0, 1, 2, …, and equally spaced grid lines parallel to OX, defined by $y_j = jk$, j = 0, 1, 2, …, as shown in Figure 1.

Denote the value of U at the representative mesh point P(ih, jk) by

$$U_p = U(ih, jk) = U_{i,j}.$$

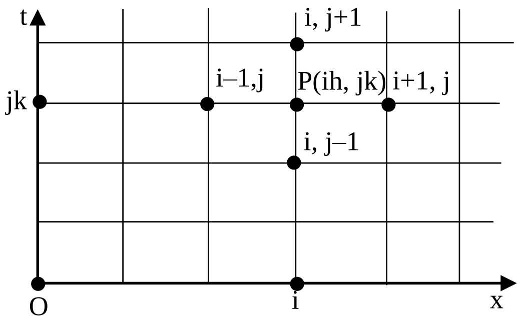

Figure 1 The x-t plane

When a function U and its derivatives are single-valued, finite and continuous functions of x, then by Taylor's theorem,

$$U(x+h) = U(x) + hU'(x) + \frac{1}{2}h^2U''(x) + \frac{1}{6}h^3U'''(x) + \dots, \quad (1)$$

And $U(x-h) =$

$$U(x) - hU'(x) + \frac{1}{2}h^2U''(x) - \frac{1}{6}h^3U'''(x) + \dots. \quad (2)$$

Addition (1) and (2), we get

$$U(x+h) + U(x-h) = 2U(x) + h^2U''(x) + O(h^4), \quad (3)$$

where $O(h^4)$ denotes terms containing fourth and higher powers of h.

Assuming these are negligible,

$$U''(x) = \frac{d^2U}{dx^2} \approx \frac{1}{h^2}\{U(x+h) - 2U(x) + U(x-h)\}. \quad (4)$$

Subtracting (2) from (1) and neglecting terms of $O(h^3)$,

$$U'(x) = \frac{dU}{dx} \approx \frac{1}{2h}\{U(x+h) - U(x-h)\}. \quad (5)$$

Equation (5) clearly approximates the slope of the tangent at P by the slope of chord AB, and is called a **central-difference approximation**. We can also approximate the slope of the chord PB, giving the **forward-difference formula**,

$$U'(x) \approx \frac{1}{h}\{U(x+h) - U(x)\} \quad (6)$$

and the slope of chord AP giving the **backward-difference formula**,

$$U'(x) \approx \frac{1}{h}\{U(x) - U(x-h)\}. \quad (7)$$

Both (6) and (7) can be written down from (1) and (2)respectively, assuming second and higher power of h are negligible. This shows that leading errors in these forward-difference and backward-difference formulae are both O(h).

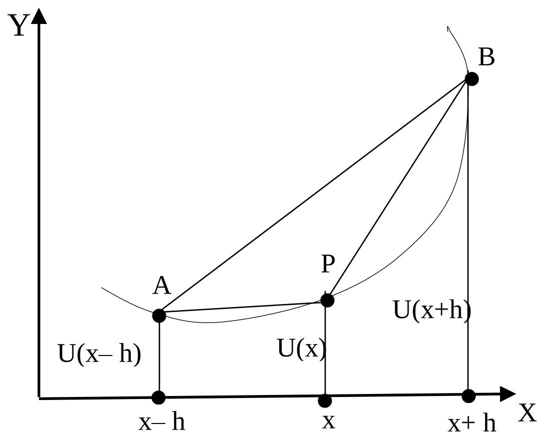

Figure 2 The slope of the tangent at P

Notation for functions of several variables

Assume U is a function of the independent variables x and t. Subdivide the x-t plane into setsofequal rectangles of sides δx = h, δt = k, by equally spaced grid lines parallel to OY, defined by x_i = ih, i = 0, 1, 2, … and equally spaced grid lines parallel to OX, defined by t_j = jk, j = 0, 1, 2, … as shown in Figure 3.

Denote the value of U at the representative mesh point P(ih, jk) by

$U_P = U(ih, jk) = U_{i,j}$. Then (4) becomes,

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_P = \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} \approx \frac{U\big((i+1)h, jk\big) - 2U\big(ih, jk\big) + U\big((i-1)h, jk\big)}{h^2},$$

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2},$$

with a leading error of $O(h^2)$. Similarly,

$$\left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j} \approx \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2},$$

with a leading error of $O(k^2)$.

The forward-difference approximation for $\frac{\partial U}{\partial t}$ at P is

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{i,j}}{k},$$

With a leading error of O(k).

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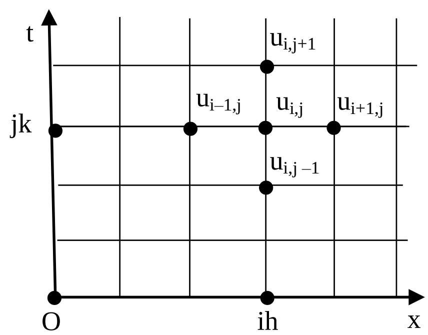

Figure 3 Equal rectangles of side δx = h, δt = k

Finite-Difference Methods

Finite-difference methods are approximate in the sense that derivatives at a point are approximated by different quotient over a small interval.

Explicit method

One finite-difference approximation to heat equation $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ is

$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$. This can be written as

$$u_{i,j+1} = r\,u_{i-1,j} + (1-2r)u_{i,j} + r\,u_{i+1,j}, \tag{8}$$

where $r = \frac{k}{h^2}$, and gives a formula (three-points formula) for the unknown temperature $u_{i,j+1}$ at the (i, j + 1)th mesh point in terms of known temperatures along the jth time-row. A method such as (11) which express one unknown pivotal value directly in terms of known pivotal values is called **Explicit method**.

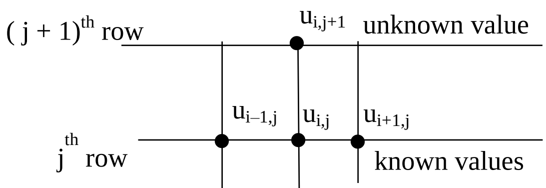

Figure 4 Explicit method (Three-points formula)

Crank-Nicolson implicit method

Crank, J. and Nicolson, P. (1947) considered the partial differential equation as being satisfied at the point $\left(ih, \left(j+\frac{1}{2}\right)k\right)$. They approximated the equation

$\left(\frac{\partial U}{\partial t}\right)_{i,j+\frac{1}{2}} = \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j+\frac{1}{2}}$ by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left\{\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right\},$$ giving

$$-r\,u_{i-1,j+1} + (2+2r)u_{i,j+1} - r\,u_{i+1,j+1} = r\,u_{i-1,j} + (2-2r)u_{i,j} + r\,u_{i+1,j} \tag{9}$$

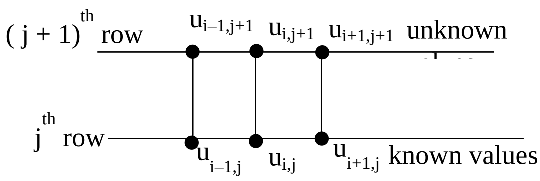

Figure 5 Crank-Nicolson implicit method

In general, the left side of (9) contains three unknown and the right side three known, pivotal values of u. If there are N internal mesh points along each time row then for j = 0 andi = 1, 2, . . ., N, equation (9) gives N simultaneous equations for the N unknown pivotal values along the first time-row in terms of known initial and boundary values. Similarly, j = 1 express N unknown values of u along the second time-row in terms of the calculated values along the first, etc. A method such as (9),where the calculation of an unknown pivotal value necessitates the solution of a set of simultaneous equations, is called a **Crank-Nicolson implicit method**.

Example (1)

As a numerical example we can solve (8) given that the ends of the rod are kept in contact with blocks of melting ice and that the initial temperature distribution in non-dimensional form is

$$\left.\begin{array}{ll}\text{(a)}\ \ U = 2x, & 0 \le x \le \frac{1}{2} \\ \text{(b)}\ \ U = 2(1-x), & \frac{1}{2} \le x \le 1.\end{array}\right\}$$

In other words, we are seeking a numerical solution of $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ which satisfies

(i) $U = 0$ at $x = 0$ and 1 for all $t > 0$. (The boundary conditions.)

(ii) $\left.\begin{array}{ll} U = 2x & 0 \le x \le \frac{1}{2} \\ \text{and}\ \ U = 2(1-x) & \frac{1}{2} \le x \le 1.\end{array}\right\}$ $t = 0$. (The initial conditions.)

For $\delta x = h = \frac{1}{10}$, the problem is symmetric with respect to $x = \frac{1}{2}$ so we need the solution only for $0 \le x \le \frac{1}{2}$.

Case I

If we take $\delta x = h = \frac{1}{10}$, $\delta t = k = \frac{1}{1000}$, so

$r = \frac{k}{h^2} = \frac{1}{10}$.

Substituting $r = \frac{1}{10}$ in (11), we get

$$u_{i,j+1} = \frac{1}{10}\left(u_{i-1,j} + 8u_{i,j} + u_{i+1,j}\right).$$

By using given conditions and Matlab programming, we get the solution as shown in Table 1.

Table 1 Solutions of Case I

		i = 0	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6
		x = 0	0.1	0.2	0.3	0.4	0.5	0.6
(j = 0)	t =0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
(j = 1)	0.001	0	0.2000	0.4000	0.6000	0.8000	0.9600	0.8000
(j = 2)	0.002	0	0.2000	0.4000	0.6000	0.7960	0.9280	0.7960
(j = 3)	0.003	0	0.2000	0.4000	0.5996	0.7896	0.9016	0.7896
(j = 4)	0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792	0.7818
(j = 5)	0.005	0	0.2000	0.3998	0.5971	0.7732	0.8597	0.7732
(j = 6)	0.006	0	0.2000	0.3996	0.5950	0.7643	0.8424	0.7643
(j = 7)	0.007	0	0.1999	0.3992	0.5924	0.7551	0.8268	0.7551
(j = 8)	0.008	0	0.1999	0.3986	0.5890	0.7460	0.8125	0.7460
(j = 9)	0.009	0	0.1998	0.3978	0.5859	0.7370	0.7992	0.7370
(j = 10)	0.010	0	0.1996	0.3968	0.5822	0.7281	0.7867	0.7281
(j = 11)	0.011	0	0.1993	0.3956	0.5783	0.7194	0.7750	0.7194
(j = 12)	0.012	0	0.1990	0.3942	0.5741	0.7108	0.7639	0.7108
(j = 13)	0.013	0	0.1986	0.3927	0.5698	0.7025	0.7533	0.7025
(j = 14)	0.014	0	0.1982	0.3910	0.5653	0.6943	0.7431	0.6943
(j = 15)	0.015	0	0.1977	0.3892	0.5608	0.6863	0 .7333	0.6863
(j = 16)	0.016	0	0.1970	0.3872	0.5562	0.6784	0.7239	0.6784
(j = 17)	0.017	0	0.1963	0.3851	0.5515	0.6708	0.7148	0.6708
(j = 18)	0.018	0	0.1956	0.3828	0.5468	0.6632	0.7060	0.6632
(j = 19)	0.019	0	0.1948	0.3805	0.5420	0.6559	0.6975	0.6559
(j = 20)	0.020	0	0.1939	0.3781	0.5373	0.6487	0.689	0.6487

Matlab program is

```
u(1,1)=0;u(2,1)=.2;u(3,1)=.4;u(4,1)=.6;
u(5,1)=.8;u(6,1)=1;u(7,1)=.8;u(8,1)=.6;
u(9,1)=.4;u(10,1)=.2;u(11,1)=0;
for j =1:31
fori =2:11
ifi<7
    u(i,j+1)=(u(i-1,j)+8*u(i,j)+u(i+1,j))/10;
elseifi==7
u(i,j+1)=u(5,j+1);
elseifi==8
u(i,j+1)=u(4,j+1);
elseifi==9
u(i,j+1)=u(3,j+1);
elseifi==10
u(i,j+1)=u(2,j+1);
elsei==11
u(i,j+1)=u(1,j+1);
end
end
end
u'
```

Case II

If we take $\delta x = h = \frac{1}{10}$, $\delta t = k = \frac{5}{1000}$, so $r = \frac{k}{h^2} = \frac{1}{2}$. Substituting $r = \frac{1}{2}$ in (11), we get

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right).$$

By using given conditions and Matlab programming, we get the solution as shown in Table 2. Matlab program is

```
u(1,1)=0;u(2,1)=.2;u(3,1)=.4;u(4,1)=.6;
u(5,1)=.8;u(6,1)=1;u(7,1)=.8;u(8,1)=.6;u(9,1)=.4;u(10,1)=.2;u(11,1)=0;
for j =1:31
fori =2:11
ifi<7
 u(i,j+1)=(u(i-1,j)+u(i+1,j))/2;
elseifi==7
u(i,j+1)=u(5,j+1);
elseifi==8
u(i,j+1)=u(4,j+1);
elseifi==9
u(i,j+1)=u(3,j+1);
elseifi==10
u(i,j+1)=u(2,j+1);
elsei==11
u(i,j+1)=u(1,j+1);
end
end
end
u'
```

Table 2 Solutions of Case II

	i= 0 x = 0	i = 1 0.1	i = 2 0.2	i = 3 0.3	i = 4 0.4	i = 5 0.5	i = 6 0.6
(j = 0)t =0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
(j = 1) 0.005	0	0.2000	0.4000	0.6000	0.8000	0.8000	0.8000
(j = 2) 0.010	0	0.2000	0.4000	0.6000	0.7000	0.8000	0.7000
(j = 3) 0.015	0	0.2000	0.4000	0.5500	0.7000	0.7000	0.7000
(j = 4) 0.020	0	0.2000	0.3750	0.5500	0.6250	0.7000	0.6250
(j = 5) 0.025	0	0.1875	0.3750	0.5000	0.6250	0.6250	0.6250
(j = 6) 0.030	0	0.1875	0.3438	0.5000	0.5625	0.6250	0.5625
(j = 7) 0.035	0	0.1719	0.3438	0.4531	0.5625	0.5625	0.5625
(j = 8) 0.040	0	0.1719	0.3125	0.453	0.5078	0.5625	0.5078
(j = 9) 0.045	0	0.1563	0.3125	0.4102	0.5078	0.5078	0.5078
(j = 10) 0.050	0	0.1563	0.2832	0.4102	0.4590	0.5078	0.4590
(j = 11) 0.055	0	0.1416	0.2832	0.3711	0.4590	0.4590	0.4590
(j = 12) 0.060	0	0.1416	0.2563	0.3711	0.4150	0.4590	0.4150
(j = 13) 0.065	0	0.1282	0.2563	0.3357	0.4150	0.4150	0.4150
(j = 14) 0.070	0	0.1282	0.2319	0.3357	0.3754	0.4150	0.3754
(j = 15) 0.075	0	0.1160	0.2319	0.3036	0.3754	0.3754	0.3754
(j = 16) 0.080	0	0.1160	0.2098	0.3036	0.3395	0.3754	0.3395
(j = 17) 0.085	0	0.1049	0.2098	0.2747	0.3395	0.3395	0.3395
(j = 18) 0.090	0	0.1049	0.1898	0.2747	0.3071	0.3395	0.3071
(j = 19) 0.095	0	0.0949	0.1898	0.2484	0.3071	0.3071	0.3071
(j = 20) 0.100	0	0.0949	0.1717	0.2484	0.2778	0.3071	0.2778

Example (2)

Consider the equation $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $0 < x < 1$, $t > 0$, where the boundary conditions and initial conditions are

(i) $U = 0$, $x = 0$ and 1, $t \geq 0$,

(ii) $U = 2x$, $0 \leq x \leq \frac{1}{2}$, $t = 0$,

(iii) $U = 2(1-x)$, $\frac{1}{2} \leq x \leq 1$, $t = 0$.

Then we can calculate a numerical solution by using the Crank-Nicolson implicit method as follow:

Take $h = \frac{1}{10}$, $r = 1$, then $k = \frac{1}{100}$. And then, $r = 1$ in (13), we get,

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \tag{10}$$

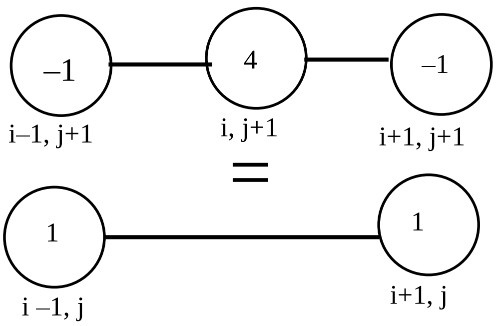

Figure 6 Crank-Nicolson implicit method

Denote $u_{i,j+1}$ by u_i (i = 1, 2, . . ., 9).

Because of symmetry, $u_6 = u_4$, $u_7 = u_3$, $u_8 = u_2$, $u_9 = u_1$, $u_{10} = u_0$.

j = 0 in (16) we get,

$-u_{i-1,1} + 4u_{i,1} - u_{i+1,1} = u_{i-1,0} + u_{i+1,0}$.

For i = 1, $-u_{0,1} + 4u_{1,1} - u_{2,1} = u_{0,0} + u_{2,0}$, then $4u_1 - u_2 = 0.4$,

for i = 2 , $-u_{1,1} + 4u_{2,1} - u_{3,1} = u_{1,0} + u_{3,0}$, then $-u_1 + 4u_2 - u_3 = 0.8$,

for i = 3, $-u_{2,1} + 4u_{3,1} - u_{4,1} = u_{2,0} + u_{4,0}$, then $-u_2 + 4u_3 - u_4 = 0.4 + 0.8 = 1.2$,

for i = 4, , $-u_3 + 4u_4 - u_5 = 0.6 + 1.0 = 1.6$,

for i = 5, $-u_4 + 4u_5 - u_6 = 0.8 + 0.8 = 1.6$.

We have, $4u_1 - u_2 = 0.4$,

$-u_1 + 4u_2 - u_3 = 0.8$, $-u_2 + 4u_3 - u_{448} = 1.210$,

$-u_3 + 4u_4 - u_5 = 1.6$, $-u_4 + 4u_5 - u_6 = 1.6$.

Then we get,

$u_1 = 0.1989$, $u_2 = 0.3956$, $u_3 = 0.5834$, $u_4 = 0.7381$, $u_5 = 0.7691$.

For second-time step, j = 1in (10) we get,

for i = 1, $-0 + 4u_1 - u_2 = 0 + 0.3956 = 0.3956$,

fori= 2, $-u_1 + 4u_2 - u_3 = 0.1989 + 0.5834 = 0.7823$,

for i = 3, $-u_2 + 4u_3 - u_4 = 0.3956 + 0.7831 = 1.1337$,

for i = 4, $-u_3 + 4u_4 - u_5 = 0.5834 + 0.7691 = 1.3525$,

for i = 5, $-u_4 + 4u_5 - u_6 = 0.7831 + 0.7831 = 1.4762$.

Thus, $u_1 = 0.1936$, $u_2 = 0.3789$, $u_3 = 0.5400$, $u_4 = 0.6461$, $u_5 = 0.6921$.

For third-time step, j = 2 in (16) we get,

for i = 1, $-0 + 4u_1 - u_2 = 0 + 0.3789 = 0.3789$,

for i = 2, $-u_1 + 4u_2 - u_3 = 0.1936 + 0.5400 = 0.7336$,

for i = 3, $-u_2 + 4u_3 - u_4 = 0.3789 + 0.6461 = 1.0250$,

for i = 4, $-u_3 + 4u_4 - u_5 = 0.5400 + 0.6921 = 1.2321$,

for i = 5,

$-u_4 + 4u_5 - u_6 = 0.6461 + 0.6461 = 1.2922$.

Thus, $u_1 = 0.1826$, $u_2 =$ 0.3515,

u_3 0.4902, $u_4 = 0.5843$, $u_5 = 0.6152$.

We get the solution of given differential equation as shown in Table 3.

Table 3 Solutions of given differential equation

	i = 0 x =0	i = 1 0.1	i = 2 0.2	i = 3 0.3	i = 4 0.4	i = 5 0.5
t = 0.0 0	0	0.2000	0.4000	0.6000	0.8000	1.0000
t = 0. 01	0	0.1989	0.3956	0.5834	0.7381	0.7691
t = 0. 02	0	0.1936	0.3789	0.5400	0.6461	0.6921
t = 0. 03	0	0.1826	0.3515	0.4902	0.5843	0.6152

Example (3)

We applied finite difference methods to the problem of the cooling of a homogeneous rod of one unit length by radiation from its ends into air at a constant temperature, the rod being at a different constant temperature initially and thermally insulated along its length, satisfying the initial condition,

$U = 1$ for $0 \leq x \leq 1$ when $t = 0$, and the boundary conditions,

$\frac{\partial U}{\partial x} = U$ at $x = 0$, for all t,

$\frac{\partial U}{\partial x} = -U$ at x = 1, for all t.

Now we will find the temperature at each time along the rod.

We can calculate a numerical solution by using an explicit method and employing central-differences for the boundary conditions.

One explicit finite-difference representation of the given equation is

$$u_{i,j+1} = u_{i,j} + r\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right), \quad (11)$$

where $r = \frac{\delta t}{(\delta x)^2}$.

Analytical solution of the partial differential equation satisfying these boundary and initial condition is

$$U = 4\sum_{n=1}^{\infty}\left\{\frac{\sec\alpha_n}{(3+4\alpha_n^2)} e^{-4\alpha_n^2 t}\cos 2\alpha_n\left(x-\frac{1}{2}\right)\right\}\ (0 < x < 1)$$

where α_n are the positive roots of $\alpha\tan\alpha = \frac{1}{2}$.

Because of symmetry,

$u_6 = u_4$, $u_7 = u_3$, $u_8 = u_2$, $u_9 = u_1$, $u_{10} = u_0$.

At x = 0 (i = 0), (17) becomes

$$u_{0,j+1} = u_{0,j} + r\left(u_{-1,j} - 2u_{0,j} + u_{1,j}\right).$$

The boundary condition at x = 0, in terms of central-differences, can be written as

$$\frac{u_{1,j} - u_{-1,j}}{2\delta x} = u_{0,j}.$$

$$u_{1,j} - u_{-1,j} = 2\delta x u_{0,j} \Rightarrow u_{-1,j} = u_{1,j} - 2\delta x u_{0,j}.$$

i = 0 in (17) we get,

$$u_{0,j+1} = u_{0,j} + r\left(u_{1,j} - 2\delta x u_{0,j} - 2u_{0,j} + u_{1,j}\right),$$

$$u_{0,j+1} = u_{0,j} + r\left[2u_{1,j} - 2(1+\delta x)u_{0,j}\right]. \quad (12)$$

Let $\delta x = 0.1$. Then at x = 1(i = 10), (11) becomes

$$u_{10,j+1} = u_{10,j} + r\left[u_{9,j} - 2u_{10,j} + u_{11,j}\right], \quad (13)$$

and the boundary condition is

$$\frac{u_{11,j} - u_{9,j}}{2\delta x} = -u_{10,j},$$

$$u_{11,j} = u_{9,j} - 2\delta x u_{10,j}.$$

Equation (13) becomes,

$$u_{10,j+1} = u_{10,j} + 2r\left[u_{9,j} - (1+2\delta x)u_{10,j}\right].$$

If we choose $r = \frac{1}{4}$,

we get $\delta t = r(\delta x)^2 = (0.25)(0.1)^2 = 0.0025$,

and (12) becomes,

$$u_{0,j+1} = u_{0,j} + 2\times\frac{1}{4}\left[u_{1,j} - (1+\delta x)u_{0,j}\right],$$

$$u_{0,j+1} = \frac{1}{2}\left(0.9u_{0,j} + u_{1,j}\right). \quad (14)$$

And also (11) becomes,

$$u_{i,j+1} = u_{i,j} + \frac{1}{4}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right),$$

$$u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right). \quad (15)$$

For the first time step, taking j = 0 in (13) we get

$$u_{0,1} = \frac{1}{2}\left(0.9\,u_{0,0} + u_{1,0}\right) = 0.95.$$

For the first-time step, taking j = 0 and i = 1, 2, 3, 4, 5 in (14) we get,

$$u_{1,1} = \frac{1}{4}\left(u_{0,0} + 2u_{1,0} + u_{2,0}\right) = \frac{4}{4} = 1,$$

$$u_{2,1} = \frac{1}{4}\left(u_{1,0} + 2u_{2,0} + u_{3,0}\right) = \frac{4}{4} = 1,$$

$$u_{3,1} = \frac{1}{4}\left(u_{2,0} + 2u_{3,0} + u_{4,0}\right) = \frac{4}{4} = 1,$$

$$u_{4,1} = \frac{1}{4}\left(u_{3,0} + 2u_{4,0} + u_{5,0}\right) = \frac{4}{4} = 1,$$

$$u_{5,1} = \frac{1}{4}\left(u_{4,0} + 2u_{5,0} + u_{6,0}\right) = \frac{4}{4} = 1.$$

For the second-time step, taking j = 1 in (14) we get

$$u_{0,2} = \frac{1}{2}\left(0.9\,u_{0,1} + u_{1,1}\right) = 0.9275.$$

For the second-time step, taking j = 1 and i =1, 2, 3, 4, 5 in (15) we get,

$$u_{1,2} = \frac{1}{4}\left(u_{0,1} + 2u_{1,1} + u_{2,1}\right) = 0.9875,$$

$$u_{2,2} = \frac{1}{4}\left(u_{1,1} + 2u_{2,1} + u_{3,1}\right) = 1,$$

$$u_{3,2} = \frac{1}{4}\left(u_{2,1} + 2u_{3,1} + u_{4,1}\right) = 1,$$

$$u_{4,2} = \frac{1}{4}\left(u_{3,1} + 2u_{4,1} + u_{5,1}\right) = 1,$$

$$u_{5,2} = \frac{1}{4}\left(u_{4,1} + 2u_{5,1} + u_{6,1}\right) = 1.$$

The solution of given equation is as below.

Table 4

	i = 0 x = 0	i = 1 0.1	i = 2 0.2	i = 3 0.3	i = 4 0.4
t = 0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
t = 0.0025	0.9500	1.0000	1.0000	1.0000	1.0000
t = 0.0050	0.9275	0.9875	1.0000	1.0000	1.0000

Conclusion

In solving second-oeder parabolic equation with initial-boundary conditions, there are various methods. Among them finite-difference method, Crank-Nicolson implicit and explicit are more reliable to get better solution.

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